Coupling ice sheet momentum and thickness evolution equations

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joint collaboration with
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Coupled Problems 2015, May 19, 2015

Center for Computing Research

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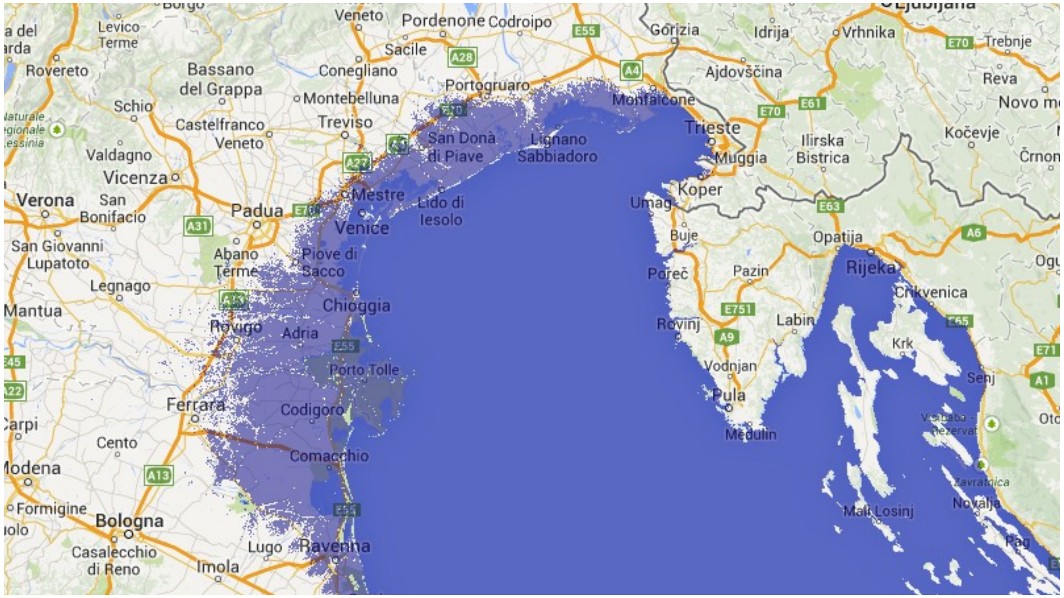
Brief introduction and motivation

- Modeling ice sheets (Greenland and Antarctica) dynamics is essential to provide estimates for sea level rise in next decades to centuries.
  Sea level rise predictions are important for policy makers.



Italy, projection for a sea levels rise of **1 m** (A.Tinlge, Google Maps/NASA)

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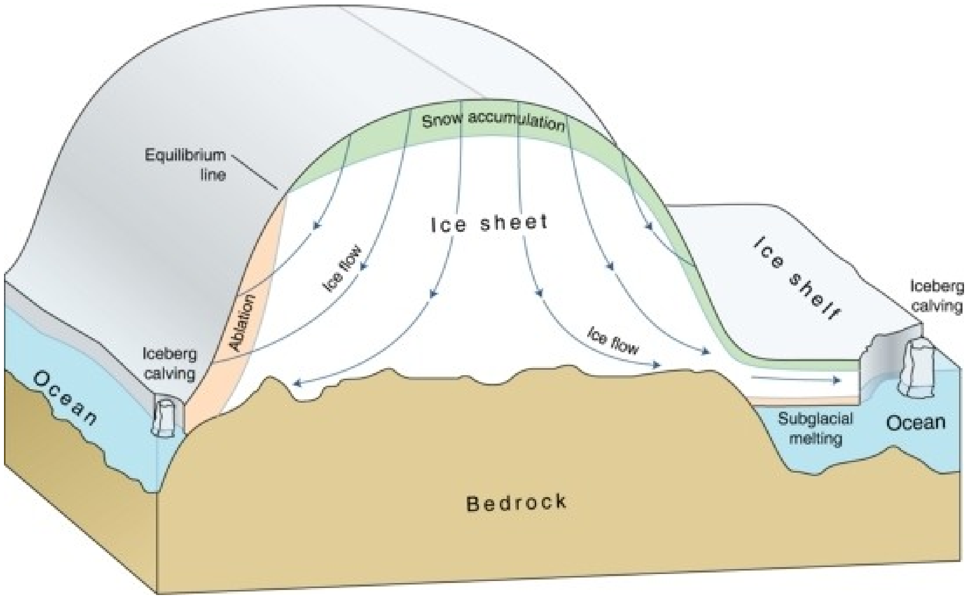
- Ice behaves like a very viscous shear-thinning fluid (similar to lava flow).



from http://www.climate.be

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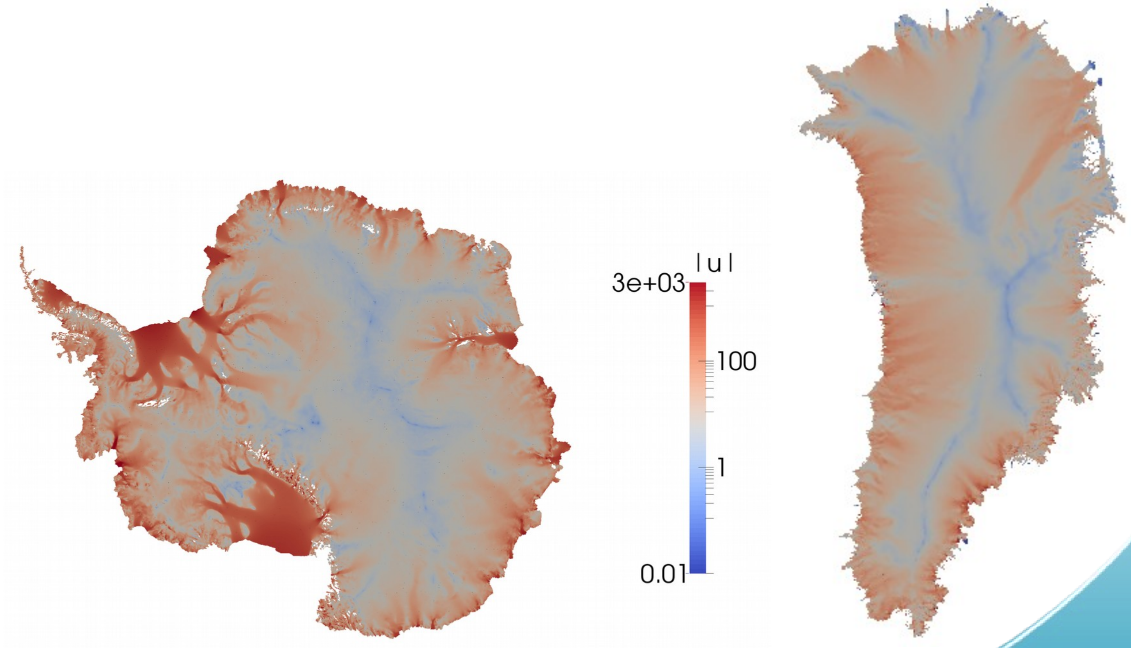
- Greenland and Antarctica ice sheets store most of the fresh water on hearth.
  They have a shallow geometry (thickness up to 3km, horizontal extensions of thousands of km).



Computed surface velocity magnitude [m/yrs]

Ice Sheet Modeling

Main components of an ice model:

- Ice flow equations (momentum and mass balance)

$$\begin{cases} -\nabla \cdot \sigma = \rho \mathbf{g} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

with:

$$\sigma = 2\mu \mathbf{D} - \Phi I, \qquad \mathbf{D}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Nonlinear viscosity:

$$\mu = \frac{1}{2}\alpha(T)\,|\mathbf{D}(\mathbf{u})|^{(p-2)}, \quad p \in (1, 2] \quad (\text{tipically } p \simeq \frac{4}{3})$$

Viscosity is singular when ice is not deforming

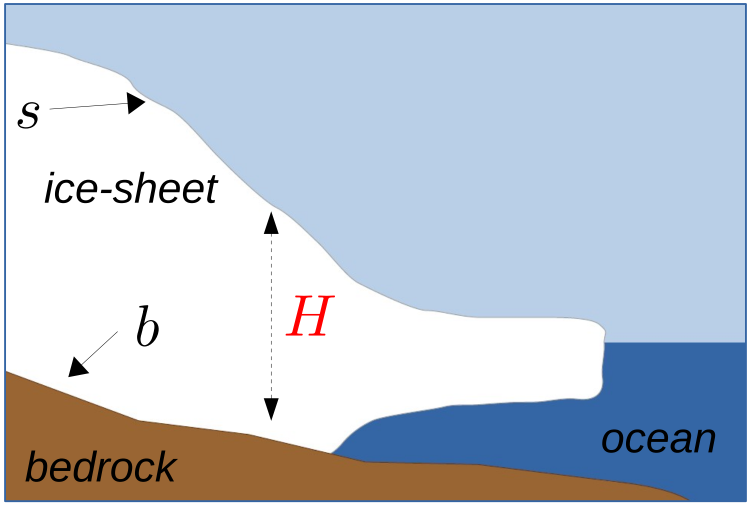
- Model for the evolution of the boundaries
(thickness evolution equation)

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\bar{\mathbf{u}}\, H) + \theta \qquad \text{with } \bar{\mathbf{u}} = \frac{1}{H}\int_z \mathbf{u}\, dz$$

Ice Sheet Modeling

Coupling

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \rho \mathbf{g} & \text{in } \Omega_H \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_H \end{cases} \qquad \frac{\partial H}{\partial t} = -\nabla \cdot (\bar{\mathbf{u}}\, H) + \theta, \quad \text{in } \Sigma$$



For grounded ice:

$$\Omega_H := \{(x, y, z) \mid z = b(x, y) + H(x, y),\ (x, y) \in \Sigma\}$$

Ice Sheet Modeling

Coupling

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \rho \mathbf{g} & \text{in } \Omega_H \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_H \end{cases} \qquad \frac{\partial H}{\partial t} = -\nabla \cdot (\bar{\mathbf{u}}\, H) + \theta, \quad \text{in } \Sigma$$

System typically coupled in a *sequential* way:

1. given H^n solve Stokes system for $\mathbf{u^n}$
2. compute $\bar{\mathbf{u}}^{\mathbf{n}}$ and solve thickness hyperbolic equation for H^{n+1}

Issue: stable only for tiny time steps.
Time steps satisfying CFL condition do NOT guarantee stability

Why? We need to simplify the equations in order to understand this.

Stokes approximations in different regimes

$$\text{Stokes}(\mathbf{u}, p) \text{ in } \Omega \in \mathbb{R}^3$$

Quasi-hydrostatic approximation

Scaling argument based on the fact that ice sheets are shallow

$$\mathbf{u} := (u,\, v,\, w)$$

$$\mathbf{D}(\mathbf{u}) = \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + \cancel{w_x}) \\ \frac{1}{2}(u_y + v_x) & v_y & \frac{1}{2}(v_z + \cancel{w_y}) \\ \cancel{\frac{1}{2}(u_z + w_x)} & \cancel{\frac{1}{2}(v_z + w_y)} & w_z \end{bmatrix}$$

$$p = \rho g(s - z) - 2\mu(u_x + v_y)$$

First Order* or Blatter-Pattyn model

$$\text{FO}(u, v) \text{ in } \Omega \in \mathbb{R}^3$$

$$-\nabla \cdot \left(2\mu\tilde{\mathbf{D}}\right) = -\rho g \nabla_{x,y}(H + b) \qquad \tilde{\mathbf{D}}(u,v) = \begin{bmatrix} 2u_x + v_y & \frac{1}{2}(u_y + v_x) & \frac{1}{2}u_z \\ \frac{1}{2}(u_y + v_x) & u_x + 2v_y & \frac{1}{2}v_z \end{bmatrix}$$

Coercive system for the horizontal components of the velocity

Dukowicz, Price and Lipscomb, 2010. J. Glaciol

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Stokes approximations in different regimes

$$\mathrm{FO}(u,v) \text{ in } \Omega \in \mathbb{R}^3$$

Ice regime:
grounded ice with frozen bed

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \frac{1}{2}u_z \\ 0 & 0 & \frac{1}{2}v_z \\ 0 & 0 & w_z \end{bmatrix}$$

$$p = \rho g(s - z)$$

Ice regime:
shelves or fast sliding grounded ice

$$\mathbf{D} = \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & 0 \\ \frac{1}{2}(u_y + v_x) & v_y & 0 \\ 0 & 0 & w_z \end{bmatrix}$$

$$p = \rho g(s - z) - 2\mu(u_x + v_y)$$

$$\mathrm{SIA}(u,v) \text{ in } \Omega \in \mathbb{R}^3$$

Shallow Ice Approximation

$$\mathrm{SSA}(u,v) \text{ in } \Sigma \in \mathbb{R}^2$$

Shallow Shelf Approximation

$$\text{Hybrid models, } \simeq SIA + SSA$$

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SIA coupled with thickness evolution

It is possible to compute the SIA solution in closed form. For constant flow rate we get

$$\begin{bmatrix} u \\ v \end{bmatrix} = C\left((s-z)^{n+1} - H^{n+1}\right)|\nabla s|^{n-1}\nabla s, \qquad (s = H + b)$$

Substituting the expression of the velocity into the thickness evolution equation*

$$\frac{\partial H}{\partial t} + \mathrm{div}\big(\eta\,\nabla H\big) = \theta - \mathrm{div}\big(\eta\,\nabla b\big), \qquad \text{with } \eta = C_1 H^2 - C_2 H^{n+2}|\nabla s|^{n-1}$$

Which is a nonlinear ***elliptic*** equation H.

In the limit case of shallow ice on frozen bedrock, the thickness evolution equation is not hyperbolic but elliptic.
We have a *diffusive CFL* condition**: $\Delta t \leq \mathrm{CFL}_{\mathrm{diff}}(\Delta x)^2$

Note: coupling the thickness evolution equation with SSA we obtain an integro-differential equation that does not feature a diffusive term.

*Fowler, ice sheets and glaciers, 1997
**Bueler and Brown, JGR, 2009

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Possible strategies to couple momentum and thickness evolution equations

1. Sequential coupling. Possibly use adaptive time steps that relies on notion of diffusive CFL, as computed using SIA approximation. **Requires many time steps.**

2. Operator splitting*: try to identify "diffusive" and "advective" parts of operator and solve the evolution equation with a IMEX scheme.

$$\mathbf{u} = \alpha \mathbf{u}_{\mathrm{SIA}} + (1-\alpha)\,\mathbf{u}_{\mathrm{SSA}}$$

Hard to identify "diffusive" part in Stokes and FO models.

3. Solve implicitly the coupling between momentum and thickness equations.

4. Use non-intrusive optimization-based coupling (see talk by M. D'Elia, afternoon session) so that momentum and thickness evolution equations can be solved in stand alone codes. **General, but possibly expensive.**

*PISM, Parallel Ice Sheet Model.

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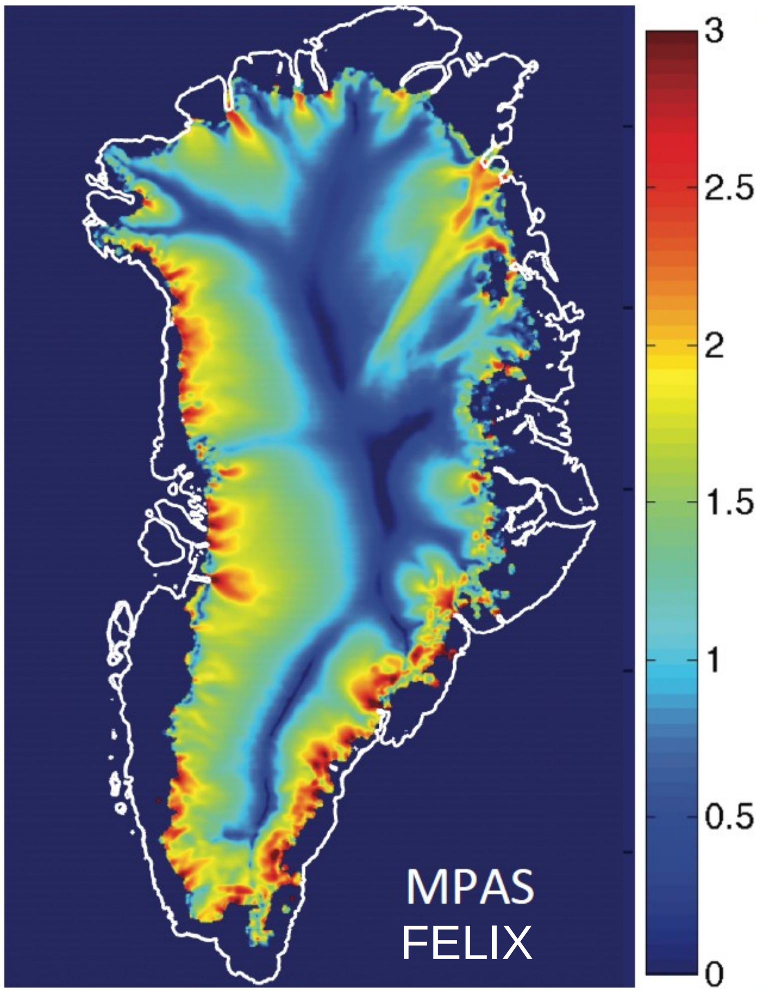
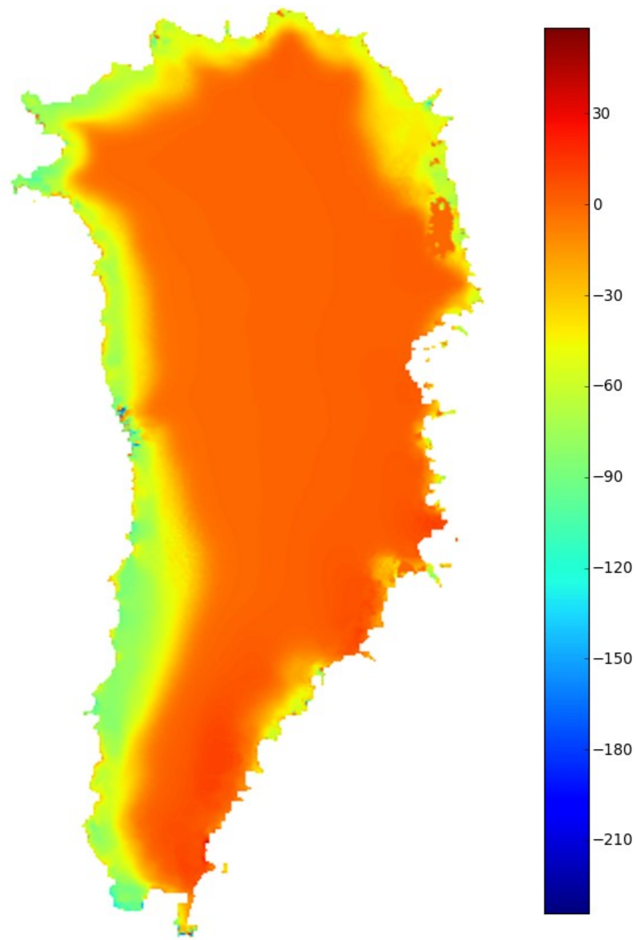
Results using sequential approach

(Ice2Sea experiment *A.J. Payne et al, PNAS 2013)*



Surface velocity magnitude [km/yr]



Thickness [m] change after 100 yr

FO eq. solved using the finite element implementation in Albany [SNL].

Evolution eq. solved using the finite volume implementation in MPAS [LANL]

Sequential approach works fine for relatively coarse meshes (here is 5km resolution). However our goal is to use resolution of 1km, 500m and in this case the approach becomes prohibitive.

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Implicit coupling of Stokes and thickness evolution equation

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}^{(n+1)}) = \rho\mathbf{g} & \text{in } \Omega_{H^{(n+1)}} \\ \nabla \cdot \mathbf{u}^{(n+1)} = 0 & \text{in } \Omega_{H^{(n+1)}} \end{cases} \qquad \frac{H^{(n+1)} - H^n}{\Delta t} + \nabla \cdot \left(\bar{\mathbf{u}}^{(n+1)} H^{(n+1)} \right) = \theta^n$$

This implicit discretization should mitigate the stability issues but is very expensive because the geometry is changing during the iterations. (Using Newton method we need to compute shape derivatives).

Idea: when using FO the thickness is exposed in the momentum equation and we may not need to to change the domain.

$$\begin{cases} -\nabla \cdot \left(\mu \tilde{\mathbf{D}} \left(\mathbf{u}^{(n+1)} \right) \right) = -\rho g \nabla \left(b + H^{(n+1)} \right) & \text{in } \Omega_{H^n} \\ \nabla \cdot \mathbf{u}^{(n+1)} = 0 & \text{in } \Omega_{H^n} \end{cases} \qquad \frac{H^{(n+1)} - H^n}{\Delta t} + \nabla \cdot \left(\bar{\mathbf{u}}^{(n+1)} H^{(n+1)} \right) = \theta^n$$

Working with external code limitations

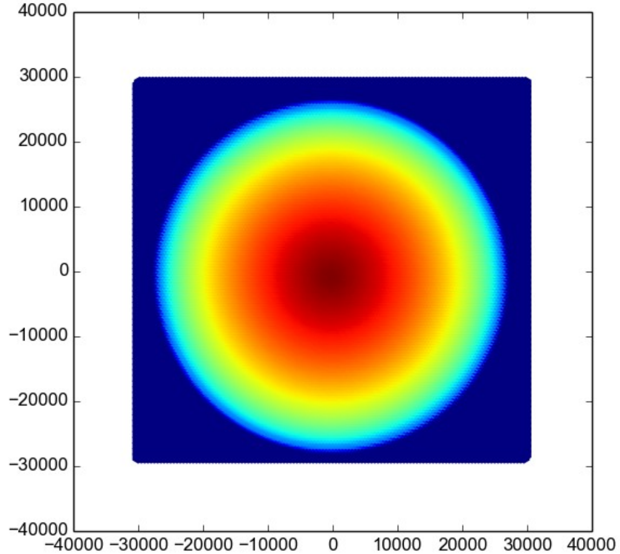
MPAS (climate library, LANL)		Albany-FELIX (finite element, SNL)
unstructured **explicit** finite volume on Voronoi grids.	$\xrightarrow{H}$	unstructured finite element method
solves for thickness equation as a tracer transported by ice velocity (upwind method).	$\xleftarrow{\mathbf{u}}$	**solves FO for velocity**

Need to improve overall solution acting only on the velocity solver.

$$\begin{cases} -\nabla \cdot \left(\mu \tilde{\mathbf{D}}\left(\mathbf{u}\right)\right) = -\rho g \nabla \left(b + H\right) & \text{in } \Omega_{H^n} \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_{H^n} \end{cases} \qquad \frac{H - H^n}{\Delta t} + \nabla \cdot \left(\bar{\mathbf{u}} H^n\right) = \theta^n$$

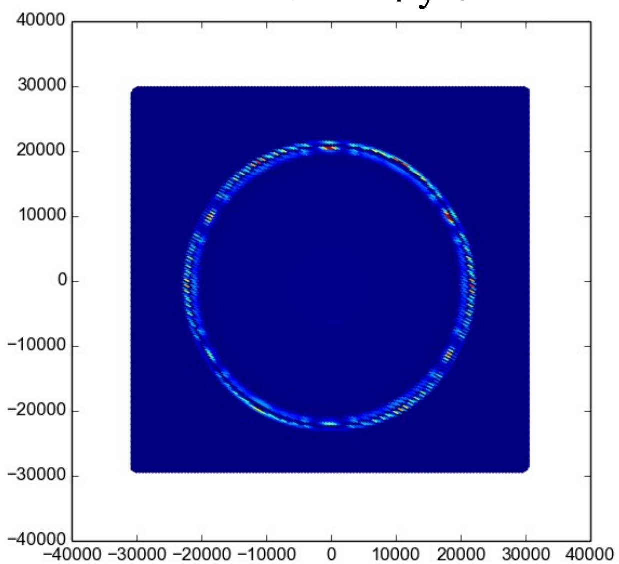
Preliminary results on dome benchmark

H at t=200 yrs



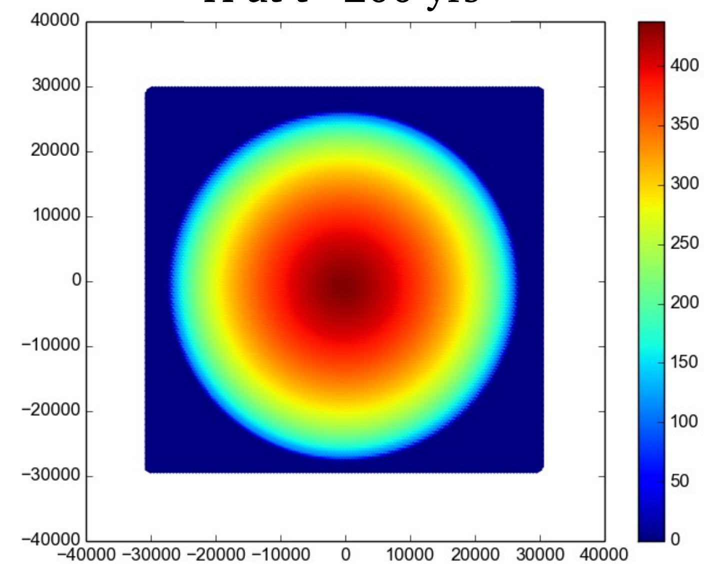
Reference solution computed with sequential approach and time step of 5 months.

H at t=4 yrs



Solution obtained with sequential coupling, **dt =1 yr**

H at t=200 yrs



Solution obtained with implicit coupling, **dt =40 yrs**

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Future development

- Investigate robustness/efficiency of the implicit method on real geometries/problem.

- Study mathematical and numerical properties of the scheme.

- Solve thickness evolution as an obstacle problem* (variational inequality) in order to avoid negative thickness and solve more accurately the margin of ice sheets.

Thank you!

*Ed Bueler, AGU meeting, 2014