Ice Sheet Dynamics: High-Order Approximation on the Sphere

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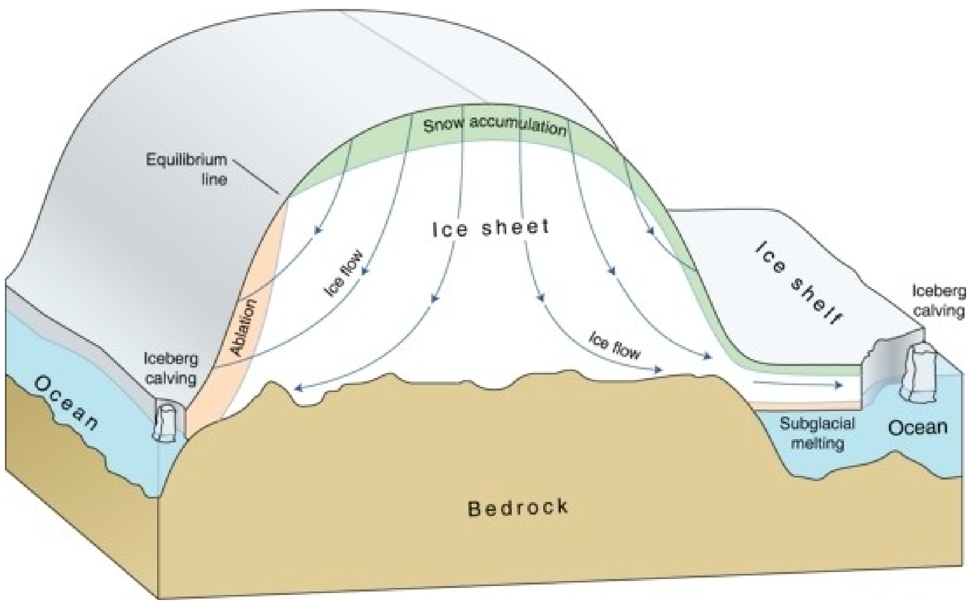
Brief introduction and motivation

- Modeling ice sheets (Greenland and Antarctica) dynamics is essential to provide estimates for sea level rise in next decades to centuries.
- Ice behaves like a very viscous shear-thinning fluid (similar to lava flow) and can be modeled with nonlinear Stokes equation.



from http://www.climate.be

Brief motivation and introduction

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- Ice behaves like a very viscous shear-thinning fluid (similar to lava flow) and can be modeled with nonlinear Stokes equation.

- Greenland and Antarctica ice sheets have a shallow geometry (thickness up to 3km, horizontal extensions of thousands of km).

- Several ice sheet models are derived relying on the fact that the domain is shallow and they handle differently horizontal coordinates (x-y) and vertical coordinate z. However, ice sheets lie on earth surface and are not planar.

- Here we investigate the effect of assuming planar geometry in approximate models.

Ice Sheet Modeling

Ice momentum equations

- **Ice flow equations** (momentum and mass balance)

$$\begin{cases} -\nabla \cdot \sigma = \rho \mathbf{g} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

with:

$$\sigma = 2\mu \mathbf{D} - pI, \qquad \mathbf{D}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Nonlinear viscosity:

$$\mu = \frac{1}{2}\alpha(T)\,|\mathbf{D}(\mathbf{u})|^{\frac{1}{n}-1}, \quad n \geq 1, \quad \text{(tipically } n \simeq 3)$$

Viscosity is singular when ice is not deforming

Stokes approximations in different regimes

$$\text{Stokes}(\mathbf{u}, p)$$

Quasi-hydrostatic approximation

Scaling argument based on the fact that ice sheets are shallow

$$\mathbf{u} := (u,\ v,\ w)$$

$$\mathbf{D}(\mathbf{u}) = \begin{bmatrix} u_x & \frac{1}{2}\left(u_y + v_x\right) & \frac{1}{2}\left(u_z + \cancel{w_x}\right) \\ \frac{1}{2}\left(u_y + v_x\right) & v_y & \frac{1}{2}\left(v_z + \cancel{w_y}\right) \\ \frac{1}{2}\left(u_z + \cancel{w_x}\right) & \frac{1}{2}\left(v_z + \cancel{w_y}\right) & w_z \end{bmatrix}$$

$$\mu = \mu(|\mathbf{D}(\mathbf{u})|)$$

3rd momentum equation

$$-\partial_z(2\mu\, w_z - p) \approx -\rho g,$$

continuity equation

$$w_z = -(u_x + v_y)$$

$$\Longrightarrow \quad p = \rho g(s - z) - 2\mu(u_x + v_y)$$

First Order* or Blatter-Pattyn model

$$\text{FO}(u, v)$$

$$-\nabla \cdot \left(2\mu\tilde{\mathbf{D}} - \rho g(s - z)\mathbf{I}\right) = 0$$

$$\tilde{\mathbf{D}}(u, v) = \begin{bmatrix} 2u_x + v_y & \frac{1}{2}\left(u_y + v_x\right) & \frac{1}{2}u_z \\ \frac{1}{2}\left(u_y + v_x\right) & u_x + 2v_y & \frac{1}{2}v_z \end{bmatrix}$$

Coercive system for the horizontal components of the velocity

**Dukowicz, Price and Lipscomb, 2010. J. Glaciol*

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Stokes approximations in different regimes

$$\mathrm{FO}(u,v)$$

Ice regime:
grounded ice with frozen bed

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \frac{1}{2}u_z \\ 0 & 0 & \frac{1}{2}v_z \\ 0 & 0 & w_z \end{bmatrix}$$

$$p = \rho g(s-z)$$

$$\mathrm{SIA}(u,v)$$

Shallow Ice Approximation

Ice regime:
shelves or fast sliding grounded ice

$$\mathbf{D} = \begin{bmatrix} u_x & \frac{1}{2}(u_y+v_x) & 0 \\ \frac{1}{2}(u_y+v_x) & v_y & 0 \\ 0 & 0 & w_z \end{bmatrix}$$

$$p = \rho g(s-z) - 2\mu(u_x+v_y)$$

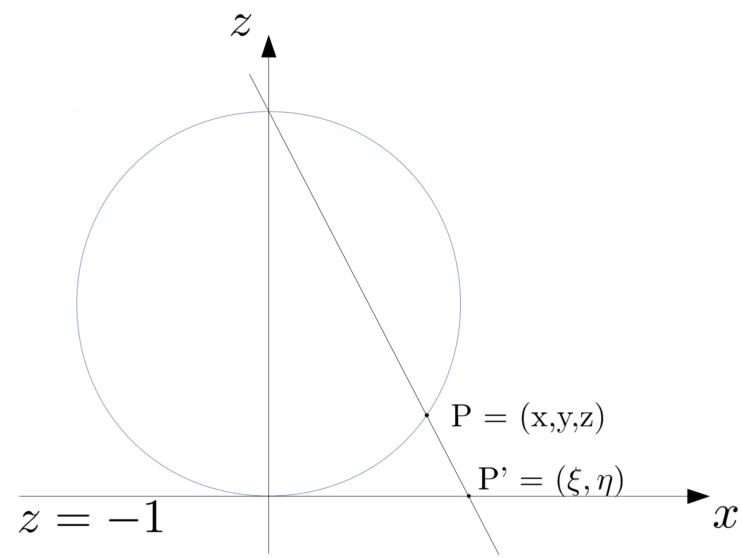
$$\mathrm{SSA}(u,v)$$

Shallow Shelf Approximation

$$\text{Hybrid models, } \simeq SIA + SSA$$

Curvilinear coordinate system

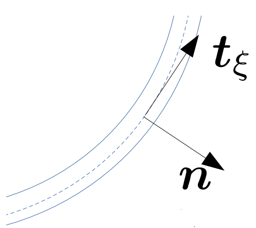
Stereographic Projection



$$\boldsymbol{x}_S = \left(\frac{\xi}{\rho(\xi,\eta)}, \ \frac{\eta}{\rho(\xi,\eta)}, \ 1 - \frac{2}{\rho(\xi,\eta)} \right)$$

$$\text{with } \rho(\xi,\eta) = 1 + \left(\frac{\xi}{2R}\right)^2 + \left(\frac{\eta}{2R}\right)^2.$$

Shell domain



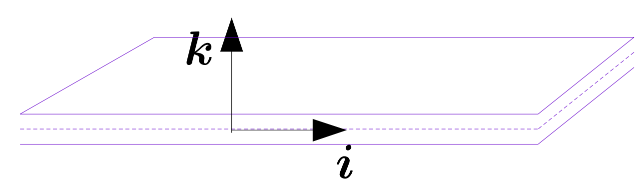
$$(\boldsymbol{t}_\xi, \boldsymbol{t}_\eta, \boldsymbol{n}) = \left(\frac{\partial x}{\partial \xi} \left| \frac{\partial x}{\partial \xi} \right|^{-1}, \ \frac{\partial x}{\partial \eta} \left| \frac{\partial x}{\partial \eta} \right|^{-1}, \ \frac{\boldsymbol{t}_\xi \times \boldsymbol{t}_\eta}{|\boldsymbol{t}_\xi \times \boldsymbol{t}_\eta|} \right)$$

$$\boldsymbol{x}(\xi,\eta,\zeta) = \boldsymbol{x}_S(\xi,\eta) + \zeta \boldsymbol{n}$$

$\boldsymbol{u}$= u $\boldsymbol{t}_\xi + v\boldsymbol{t}_\eta + w\boldsymbol{n}$

$\boldsymbol{g}$= -g $\boldsymbol{n}$

Extruded planar domain



$\boldsymbol{u}$= u $\boldsymbol{i}$ + v $\boldsymbol{j}$+ w $\boldsymbol{k}$

$\boldsymbol{g}$= -g $\boldsymbol{k}$

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Derivation of First Order model in curvilinear coordinate system

Metric tensor: $$\mathrm{g}_{ij} = \begin{bmatrix} h^2 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with $$\mathrm{h} = \left(1 + \frac{\zeta}{R}\right)\frac{1}{\rho(\xi,\eta)} \approx \frac{1}{\rho(\xi,\eta)} \qquad (\zeta \ll R)$$

and $$\rho(\xi,\eta) = 1 + \left(\frac{\xi}{2R}\right)^2 + \left(\frac{\eta}{2R}\right)^2$$

FO approximation of strain rate tensor

$$\mathbf{D}(\mathbf{u}) = \begin{bmatrix} \frac{1}{h}u_\xi + \frac{1}{h^2}h_\eta v & \frac{1}{2h}\left(u_\eta - \frac{h_\eta}{h}u + v_\xi - \frac{h_\xi}{h}v\right) & \frac{1}{2}\left(u_\zeta + \cancel{\frac{1}{h}w_\xi}\right) \\ \frac{1}{2h}\left(u_\eta - \frac{h_\eta}{h}u + v_\xi - \frac{h_\xi}{h}v\right) & \frac{1}{h}v_\eta + \frac{1}{h^2}h_\xi u & \frac{1}{2}\left(v_\zeta + \cancel{\frac{1}{h}w_\eta}\right) \\ \frac{1}{2}\left(u_\zeta + \cancel{\frac{1}{h}w_\xi}\right) & \frac{1}{2}\left(v_\zeta + \cancel{\frac{1}{h}w_\eta}\right) & w_\zeta \end{bmatrix}$$

$$\mu = \mu(|\mathbf{D}(\mathbf{u})|)$$

3rd momentum equation

$$-\,\partial_\zeta(2\mu\, w_\zeta - p) = -\rho g,$$

Continuity equation

$$w_\zeta = -\left(\frac{1}{h}u_\xi + \frac{1}{h^2}h_\eta v + \frac{1}{h}v_\eta + \frac{1}{h^2}h_\xi u\right)$$

$$\Longrightarrow\ p = \rho g(s - \zeta) - 2\mu\left(\frac{1}{h}u_\xi + \frac{1}{h^2}h_\eta v + \frac{1}{h}v_\eta + \frac{1}{h^2}h_\xi u\right)$$

First Order model on sphere

$$\begin{cases} -\nabla \cdot \left(2\mu\tilde{\mathbf{D}}(u,v) - \rho g(s-\zeta)\mathbf{I}\right) = \mathbf{0} & \text{in } \Omega \\ -2\mu\tilde{\mathbf{D}}(u,v)\boldsymbol{n} + \beta[u,v]^T = \mathbf{0} & \text{on } \Sigma \end{cases}$$

with

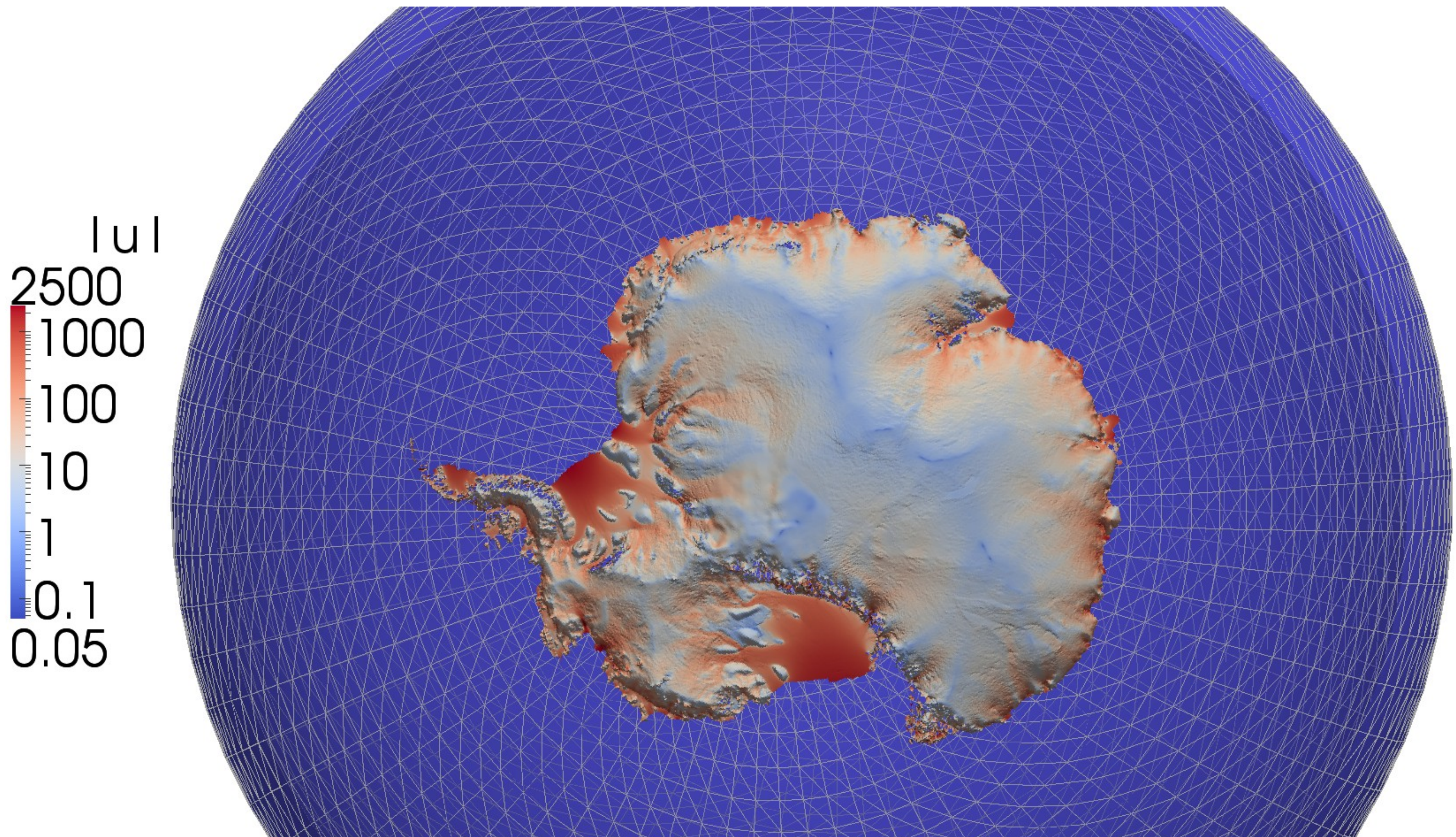
$$\tilde{\mathbf{D}}(u,v) = \begin{bmatrix} \frac{2}{h}u_\xi + \frac{2}{h^2}h_\eta v + \frac{1}{h}v_\eta + \frac{1}{h^2}h_\xi u & \frac{1}{2h}\left(u_\eta - \frac{h_\eta}{h}u + v_\xi - \frac{h_\xi}{h}v\right) & \frac{1}{2}u_\zeta \\ \frac{1}{2h}\left(u_\eta - \frac{h_\eta}{h}u + v_\xi - \frac{h_\xi}{h}v\right) & \frac{2}{h}v_y + \frac{2}{h^2}h_\xi u + \frac{1}{h}u_\eta + \frac{1}{h^2}h_\eta v & \frac{1}{2}v_\zeta \end{bmatrix}$$

Weak formulation

$$\int_\Omega (2\mu\mathbf{D}(u,v) - \rho g(s-\zeta)\mathbf{I}) : \mathbf{D}(\phi_1,\phi_2)\; h^2 d\xi d\eta d\zeta + \int_\Sigma \beta(u\phi_1 + v\phi_2)\; h^2 d\xi d\eta = 0$$

Surface velocity magnitude of Antarctic ice sheet computed with FO on the sphere

Surface velocity magnitude [m/yr], ice sheet thickness not at scale (100 X)



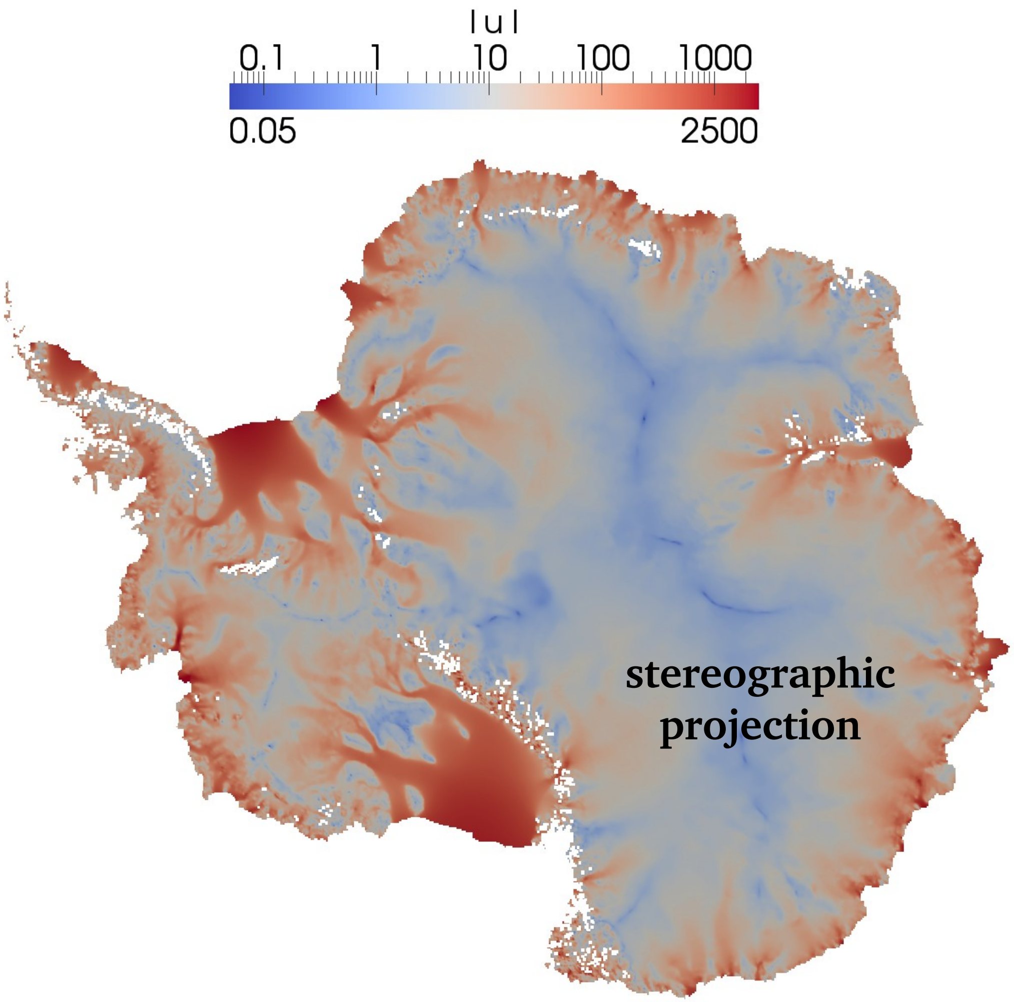
Gometry (Cornford, Martin et al.,2014)

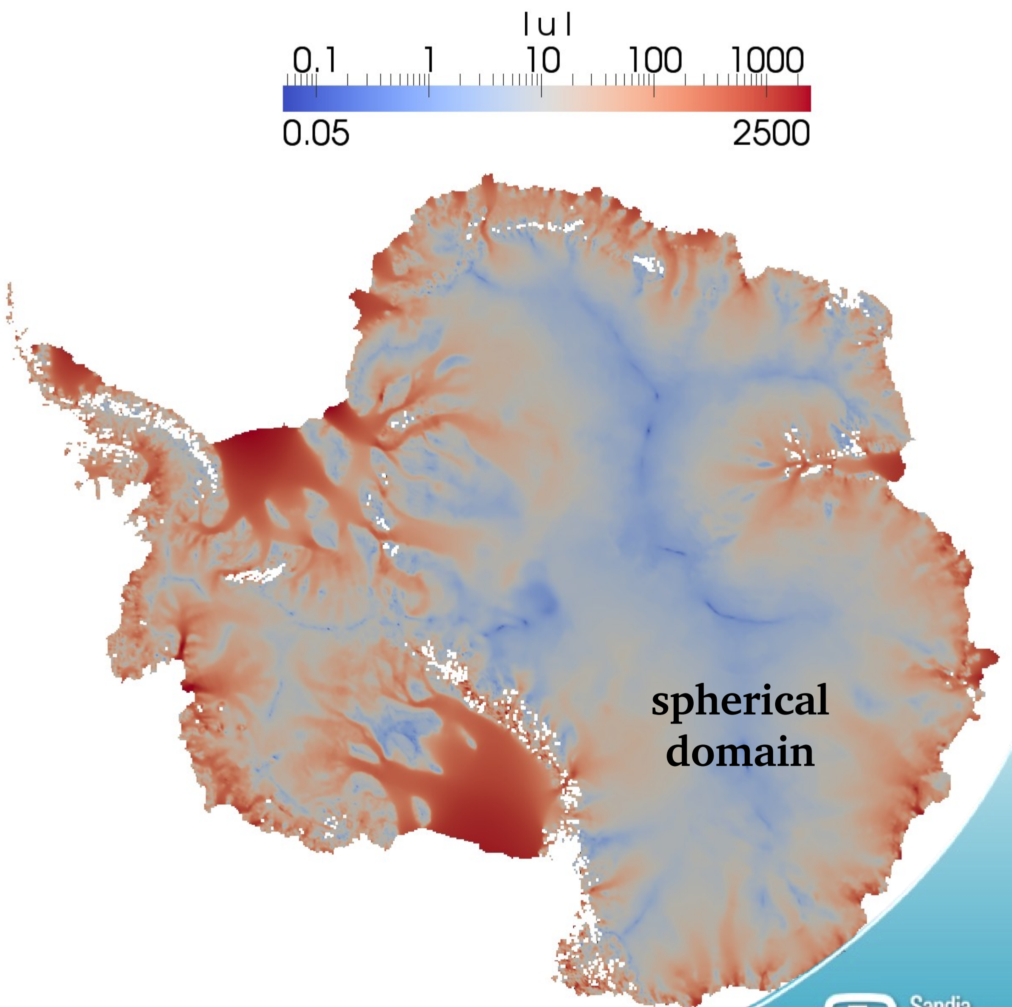
Bedmap2 (Fretwell et al., 2013)

Temperature (Pattyn, 2010)

Comparison of computed surface velocities: planar stereographic projection vs spherical domain

Spot the difference!

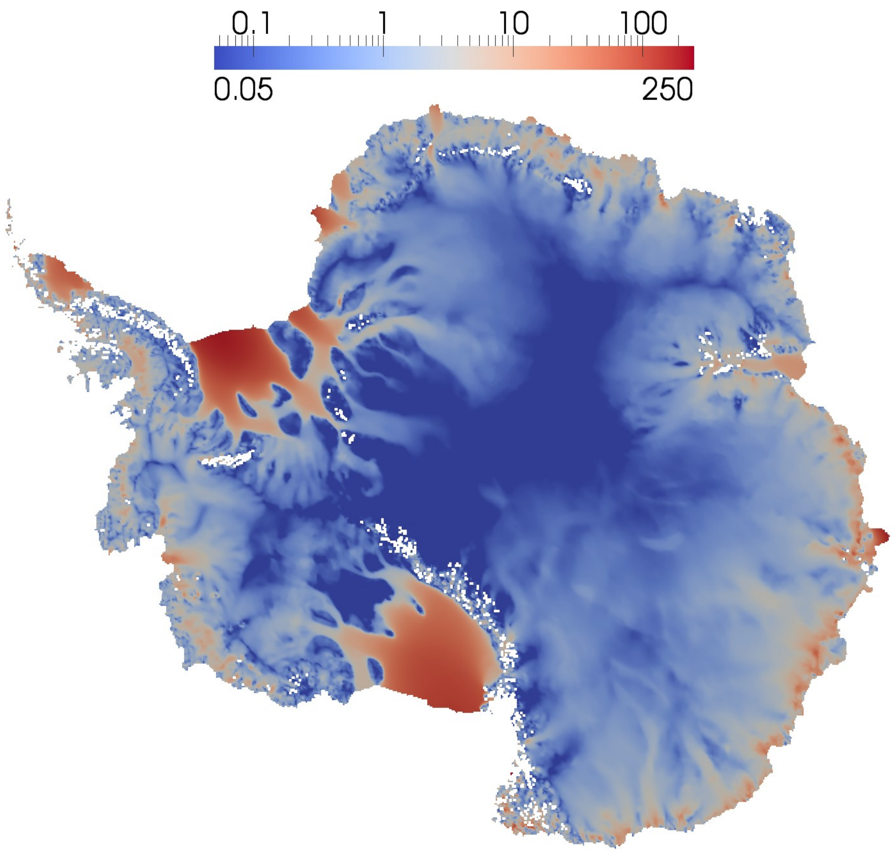




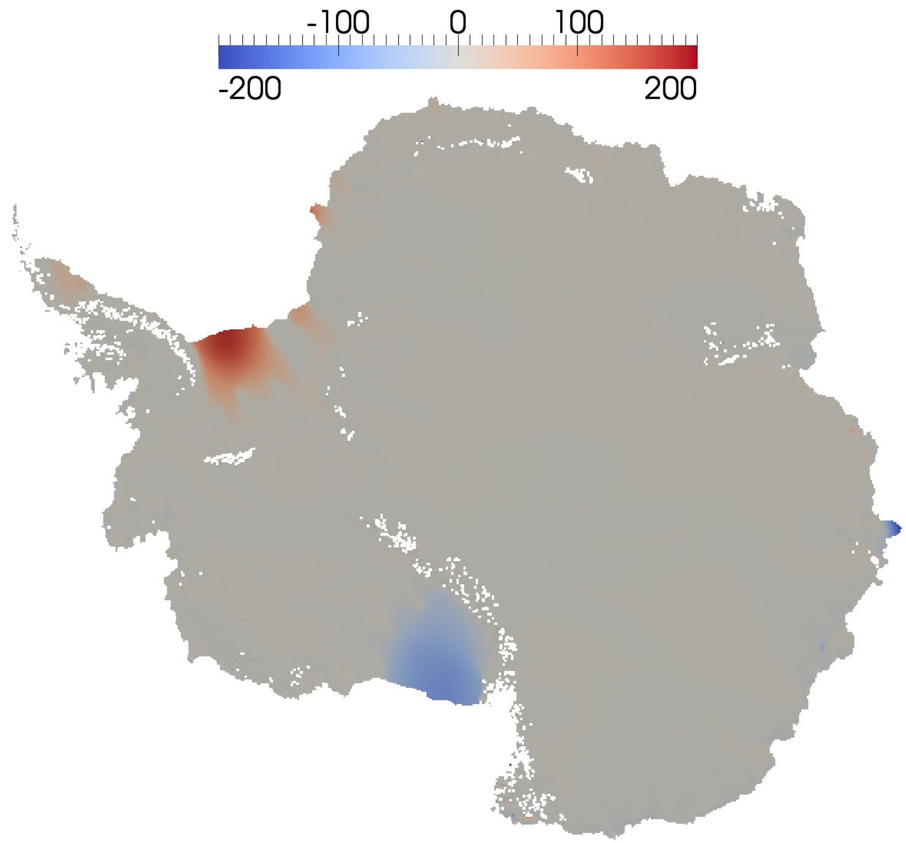
Surface velocity magnitude [m/yr]

Difference between surface velocities computed using spherical FO and planar FO

magnitude of surface velocity difference [m/yr]



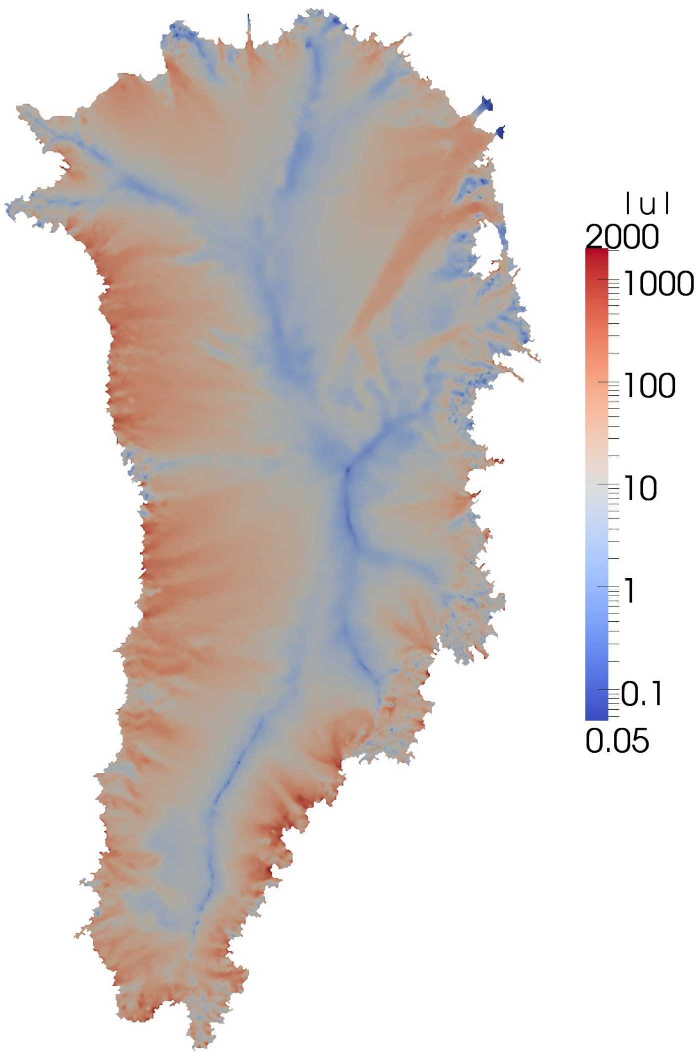
v component of surface velocity difference [m/yr]



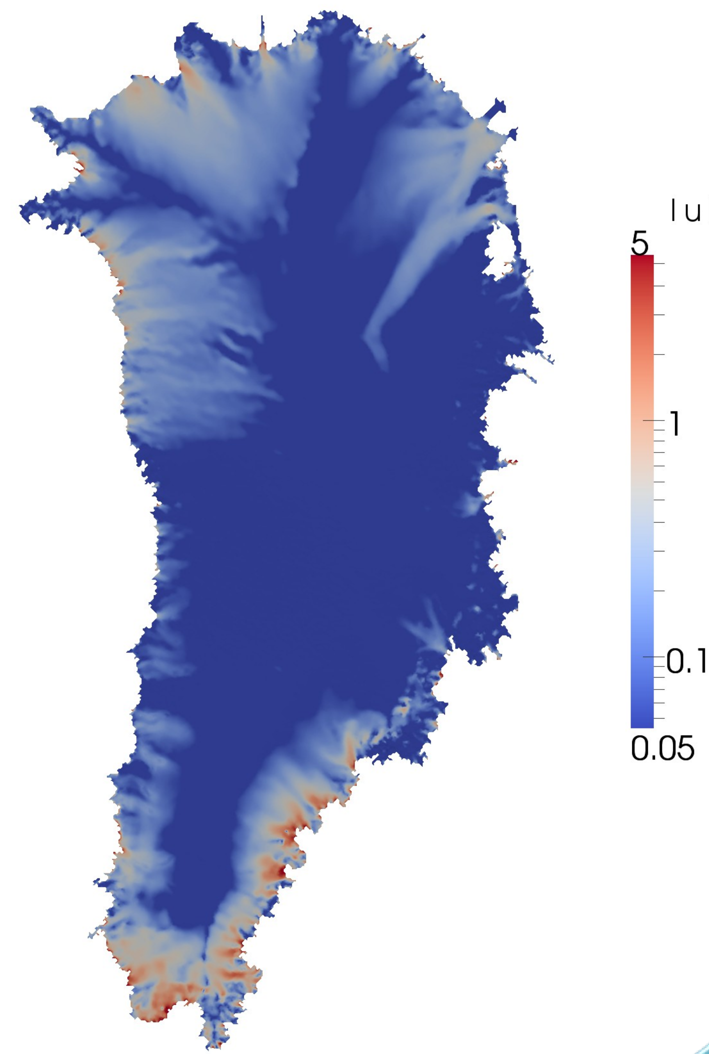
Relative difference in the velocities is at most 10% in fast flow areas.

Comparison of spherical FO planar FO on Greenland ice sheet

Surface velocity magnitude [m/yr]
(spherical FO)



magnitude of surface velocity difference [m/yr]



Relative difference in the velocities are below 1%.

Geometry and fields Bamber et al.[2013], temperature computed with CISM (Shannon et al. [2013])

Conclusions

- In principle, approximation of Stokes equations should be derived considering the curvature of earth surface rather than assuming a planar shell.
- Even for Antarctica ice sheet, differences between the standard FO model and the spherical one are modest but maybe not negligible.
- Need to compare the planar and nonplanar approximations in an evolution study and validate the approximations with Stokes model.

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